

The effective exponent $\gamma(Q)$ and the slope of the β function

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Abstract:

The slope of the β function at a fixed point is commonly thought to be RG invariant and to be the critical exponent γ^* that governs the approach of any physical quantity \mathcal{R} to its fixed-point limit: $\mathcal{R}^* - \mathcal{R} \propto Q^{\gamma^*}$. Chýla has shown that this is not quite true. Here we define a proper RG invariant, the “effective exponent” $\gamma(Q)$, whose fixed-point limit is the true γ^* .

The β function, $\beta(a) \equiv \mu \frac{da}{d\mu}$, of a renormalizable quantum field theory is renormalization-scheme (RS) dependent. The slope of the β function at a fixed point, however, is commonly believed to be scheme invariant. That is not quite true.

The traditional argument [1, 2] goes as follows. Consider two RS's, primed and unprimed, whose renormalized coupling constants (couplants) are related by a general scheme transformation

$$a' = a(1 + v_1 a + v_2 a^2 + \dots). \quad (1)$$

Their β functions are related by

$$\beta'(a') = \frac{da'}{da} \beta(a). \quad (2)$$

If $\beta(a)$ vanishes at $a = a^*$ then $\beta'(a')$ will vanish at the corresponding $a' = a'^*$. (The scheme transformation could push a'^* off to infinity, but let us assume that both a^* and a'^* exist and are finite.) The derivative of the β function will transform as

$$\frac{d\beta'}{da'} = \frac{d\beta}{da} + \beta(a) \frac{d^2 a'}{da^2} \bigg/ \frac{da'}{da}. \quad (3)$$

Since $\beta(a)$ vanishes at the fixed point, it would seem that

$$\left. \frac{d\beta'}{da'} \right|_* = \left. \frac{d\beta}{da} \right|_* \quad (\text{not really true}). \quad (4)$$

Refs. [1, 2] properly qualify this result with the proviso that da'/da must not vanish and $d^2 a'/da^2$ must not be singular, at $a = a^*$, so no criticism of these august authors is warranted. Their unwary readers, however, may get the impression that these restrictions only refer to pathological or exceptionally rare RS transformations. Chýla [3] provides a salutary corrective to that attitude. Indeed, a stark contradiction arises from trusting Eq. (4), as we discuss in an appendix below.

Here we define the “effective exponent” $\gamma(Q)$, a Q -dependent “scaling dimension” associated with a specific physical quantity \mathcal{R} . It is related to the slope of the β function but has an extra term that is crucial for its RS invariance. Our discussion will be at the formal level, except for some brief comments at the end.

Consider some physical quantity \mathcal{R} , which may depend on several experimentally defined parameters. We may always single out one such parameter, “ Q ,” with dimensions of energy, and make all other parameters dimensionless. (The precise definition of Q in any specific case is left to the reader.) For definiteness we assume that the theory is asymptotically free as $Q \rightarrow \infty$, though our results are easily adaptable to the opposite case. Also for definiteness we assume that \mathcal{R} has a perturbation expansion

$$\mathcal{R} = a^P (1 + r_1 a + \dots), \quad (5)$$

although our key results apply whether or not \mathcal{R} is calculated (or even calculable) perturbatively.

Since \mathcal{R} is a physical quantity and Q is a physical parameter, the successive logarithmic derivatives of \mathcal{R} :

$$\mathcal{R}_{[n+1]} \equiv Q \frac{d\mathcal{R}_{[n]}}{dQ} \quad (6)$$

for $n = 1, 2, 3, \dots$, with $\mathcal{R}_{[1]} \equiv \mathcal{R}$, must be RS-invariant quantities, for any Q . In particular, the combination

$$\gamma(Q) \equiv \frac{\mathcal{R}_{[3]}}{\mathcal{R}_{[2]}} = 1 + Q \frac{d^2\mathcal{R}}{dQ^2} / \frac{d\mathcal{R}}{dQ} \quad (7)$$

is RS invariant. It is the exponent of the local-power-law form of $\mathcal{R}(Q)$ in the following sense: Take the first three terms of the Taylor expansion of \mathcal{R} about $Q = Q_0$ and fit them to the power-law form

$$\mathcal{R} \approx K + CQ^\gamma \quad (8)$$

to find

$$\begin{aligned} \mathcal{R}_0 &\equiv \mathcal{R}|_{Q=Q_0} = K + CQ_0^\gamma, \\ \mathcal{R}'_0 &\equiv \left. \frac{d\mathcal{R}}{dQ} \right|_{Q=Q_0} = \gamma CQ_0^{\gamma-1}, \\ \mathcal{R}''_0 &\equiv \left. \frac{d^2\mathcal{R}}{dQ^2} \right|_{Q=Q_0} = \gamma(\gamma-1)CQ_0^{\gamma-2}. \end{aligned} \quad (9)$$

These algebraic equations can be inverted to find the three parameters K, C , and γ . (Note that K is not \mathcal{R}_0 in general, though it is when $Q_0 \rightarrow 0$, assuming $\gamma > 0$.) In particular,

$$\gamma = 1 + Q_0 \frac{\mathcal{R}''_0}{\mathcal{R}'_0}, \quad (10)$$

which is the $\gamma(Q_0)$ of Eq. (7).

At high energies, where $\mathcal{R} \propto (1/\ln Q)^P$, one has a negative γ , but as Q is lowered γ becomes positive. As $Q \rightarrow 0$ it becomes the critical exponent γ^* governing the approach of \mathcal{R} to its fixed-point value \mathcal{R}^* :

$$(\mathcal{R}^* - \mathcal{R}) \propto Q^{\gamma^*} \quad \text{as } Q \rightarrow 0. \quad (11)$$

In the perturbative expansion of \mathcal{R} , in some specific RS with renormalization scale μ , the only Q dependence resides in the series coefficients r_i . For dimensional reasons, these can only depend on Q through the ratio Q/μ . Thus, we have

$$Q \frac{d\mathcal{R}}{dQ} = -\mu \left. \frac{\partial \mathcal{R}}{\partial \mu} \right|_a, \quad (12)$$

where the μ partial derivative is taken holding a constant. The total μ derivative of \mathcal{R} vanishes:

$$\mu \frac{d\mathcal{R}}{d\mu} = \mu \left. \frac{\partial \mathcal{R}}{\partial \mu} \right|_a + \beta(a) \frac{d\mathcal{R}}{da} = 0. \quad (13)$$

This RG equation says that the μ dependence of the coefficients is cancelled by the μ dependence via the couplant a . The two preceding equations lead to

$$\mathcal{R}_{[2]} \equiv Q \frac{d\mathcal{R}}{dQ} = \beta(a) \frac{d\mathcal{R}}{da}. \quad (14)$$

Since $\mathcal{R}_{[2]}$ is itself a physical quantity we can apply the same argument to it to get

$$\mathcal{R}_{[3]} = \beta(a) \frac{d\mathcal{R}_{[2]}}{da} = \beta(a) \left(\frac{d\beta}{da} \frac{d\mathcal{R}}{da} + \beta(a) \frac{d^2\mathcal{R}}{da^2} \right). \quad (15)$$

Dividing Eq. (15) by Eq. (14) yields our key result

$$\gamma(Q) = \frac{d\beta}{da} + \beta(a) \frac{d^2\mathcal{R}}{da^2} \bigg/ \frac{d\mathcal{R}}{da}. \quad (16)$$

[We digress briefly to recall a similar point made early in Ref. [4]. The anomalous dimension of a Green's function \mathcal{G} is conventionally defined as

$$\gamma_{(\mathcal{G})} \equiv \frac{\mu}{\mathcal{G}} \frac{d\mathcal{G}}{d\mu} = \frac{1}{\mathcal{G}} \left(\mu \frac{\partial \mathcal{G}}{\partial \mu} \bigg|_a + \beta(a) \frac{d\mathcal{G}}{da} \right), \quad (17)$$

which corresponds to the Callan-Symanzik equation [5] for \mathcal{G} . It is *not* a physical quantity. However, a physical quantity, an “effective exponent” for \mathcal{G} , can be defined as

$$\mathcal{R}_{(\mathcal{G})} \equiv - \frac{\lambda}{\mathcal{G}} \frac{d}{d\lambda} \mathcal{G}(\lambda p_i, \mu, a(\mu)) \bigg|_{\lambda=1}. \quad (18)$$

(It could be written as $-\frac{Q}{\mathcal{G}} \frac{d\mathcal{G}}{dQ}$, given our convention that Q is the only dimensional physical parameter with all other parameters rendered dimensionless; e.g. $Q = p_1$ with the other parameters being $p_2/p_1, \dots$) The important point here is that the wavefunction-renormalization constant $Z_{(\mathcal{G})}$ that multiplicatively renormalizes \mathcal{G} is independent of the momentum arguments p_i and cancels out in Eq. (18). By the argument leading to Eq. (12), we see that

$$\mathcal{R}_{(\mathcal{G})} = \gamma_{(\mathcal{G})} - \frac{\beta(a)}{\mathcal{G}} \frac{d\mathcal{G}}{da}, \quad (19)$$

which is analogous to Eq. (16).]

Returning to $\gamma(Q)$, it is instructive to check directly that Eq. (16) is invariant under scheme transformations. The derivatives of \mathcal{R} transform as

$$\begin{aligned} \frac{d\mathcal{R}}{da'} &= \frac{d\mathcal{R}}{da} \bigg/ \frac{da'}{da}, \\ \frac{d^2\mathcal{R}}{da'^2} &= \frac{d}{da} \left(\frac{d\mathcal{R}}{da} \bigg/ \frac{da'}{da} \right) \bigg/ \frac{da'}{da} \\ &= \left(\frac{d^2\mathcal{R}}{da^2} - \frac{d\mathcal{R}}{da} \frac{d^2a'}{da^2} \bigg/ \frac{da'}{da} \right) \frac{1}{\left(\frac{da'}{da} \right)^2}. \end{aligned} \quad (20)$$

Hence, the second term in Eq. (16) transforms as

$$\beta'(a') \frac{d^2 \mathcal{R}}{da'^2} \Big/ \frac{d\mathcal{R}}{da'} = \beta(a) \frac{d^2 \mathcal{R}}{da^2} \Big/ \frac{d\mathcal{R}}{da} - \beta(a) \frac{d^2 a'}{da^2} \Big/ \frac{da'}{da}. \quad (21)$$

Adding this to Eq. (3) we see that

$$\frac{d\beta'}{da'} + \beta'(a') \frac{d^2 \mathcal{R}}{da'^2} \Big/ \frac{d\mathcal{R}}{da'} = \frac{d\beta}{da} + \beta(a) \frac{d^2 \mathcal{R}}{da^2} \Big/ \frac{d\mathcal{R}}{da}, \quad (22)$$

confirming that $\gamma(Q)$ is genuinely scheme independent.

Further insight into $\gamma(Q)$ is the following observation. Specialize to the case $P = 1$ (or define $\mathcal{R}_{\text{new}} = \mathcal{R}_{\text{old}}^{1/P}$) and consider the “effective charge” (EC) renormalization scheme [6] defined so that $\mathcal{R} = a(1 + 0 + 0 + \dots)$. In this scheme $d^2 \mathcal{R}/da^2 = 0$, so Eq. (16) reduces to

$$\gamma(Q) = \frac{d\beta_{\text{EC}}(\mathcal{R})}{d\mathcal{R}}. \quad (23)$$

Thus $\gamma(Q)$, at any Q , is the slope of the EC β function at the corresponding \mathcal{R} . In particular, in the infrared limit, the critical exponent γ^* is the derivative of the EC β function at the fixed point. Moreover, from Eq. (16), we can say that γ^* is the derivative of the β function at the fixed point in any scheme for which $\frac{d\mathcal{R}}{da}$ is non-zero and $\frac{d^2 \mathcal{R}}{da^2}$ is non-singular at $a = a^*$. That includes a large class of possible RS’s, but by no means is this “almost all” schemes [3]. In general we must go back to Eq. (16) and carefully consider its infrared limit. A similar point applies to Eq. (19). For an instance where this subtlety arises see Ref. [7].

An important open question concerns the “universality,” or otherwise, of γ^* . Is it the same for all perturbative physical quantities \mathcal{R} ? The question hinges on whether the EC couplants a and a' for two different physical quantities \mathcal{R} and \mathcal{R}' always have $da'/da|_*$ non-zero and $d^2 a'/da^2|_*$ non-singular. Possibly yes, but it may well be that physical quantities segregate into distinct classes, each with a characteristic value of γ^* .

The preceding discussion has been entirely at the formal level. In practice, of course, one uses some approximation to \mathcal{R} and to $\beta(a)$. A whole set of other issues then arises. While physical quantities are scheme independent, perturbative approximations to them are not; scheme choice matters. Fixed points can be made to appear or disappear under scheme transformations [8, 4] when $\beta(a)$ and $\beta'(a')$ are each truncated and Eq. (2) is satisfied only up to missing higher-order terms. In the $\overline{\text{MS}}$ scheme for QCD there is no fixed point at low n_f , but this may be entirely misleading. In the EC scheme, or when the scheme choice is “optimized” [9], one finds fixed-point behaviour for $\mathcal{R}_{e^+e^-}$ in both third [10] and fourth [11] order.

Other issues beyond the formal level are the related ones of perturbation-series divergence and power-suppressed non-perturbative terms, exponentially small in the couplant.

When approximating $\gamma(Q)$, or its infrared limit γ^* , the most meaningful result comes from its original definition, Eq. (7), with \mathcal{R} replaced by its approximation. For some schemes this is the same as using Eq. (16) with the \mathcal{R} and the β function replaced by their approximations, but in other schemes this may not be the case.

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Appendix

We show here that a stark contradiction arises if the slope of the β function at the fixed point,

$$\gamma_{\text{false}}^* \equiv \left. \frac{d\beta}{da} \right|_*, \quad (24)$$

is taken to be a scheme-invariant quantity. Writing the β function as

$$\beta(a) = -ba^2 \sum_i c_i a^i \quad (25)$$

(with $c_0 \equiv 1$ and $c_1 \equiv c$), the coefficients b and c are scheme invariant, but the c_j 's (for $j = 2, 3, \dots$) are not. These c_j 's, together with $\tau = b \ln(\mu/\tilde{\Lambda})$, can serve to parametrize the renormalization-scheme dependence [9]. (Since τ goes to $-\infty$ at the fixed point, it plays no role in our discussion here.) Physical quantities are independent of the c_j 's (for $j = 2, 3, \dots$), due to cancellation between the c_j dependences of the perturbative coefficients and those of the couplant a . The c_j dependences of the couplant, $\partial a / \partial c_j$ with τ and the other c_i 's held constant, are given by functions $\beta_j(a)$ defined in [9]. In the fixed-point limit these tend to [4]

$$\frac{\partial a^*}{\partial c_j} = \frac{ba^{*j+2}}{\gamma_{\text{false}}^*}. \quad (26)$$

This result follows easily by asking how the root a^* of the equation $\sum_i c_i a^{*i} = 0$ changes as one specific c_j is varied [4]. Equivalently, if we define $B(a) \equiv \sum_i c_i a^i$, then $B^* \equiv B(a^*)$ is trivially RS invariant since it is zero in all schemes, so that

$$\left. \frac{\partial B^*}{\partial c_j} \right|_{a^*} + \frac{\partial a^*}{\partial c_j} \frac{dB^*}{da^*} = 0, \quad (27)$$

which leads directly to Eq. (26).

From Eq. (24) and $B^* = 0$ we have

$$\gamma_{\text{false}}^* = -b \sum_i i c_i a^{*i+1}. \quad (28)$$

If γ_{false}^* were a physical quantity then we would have

$$\left. \frac{\partial \gamma_{\text{false}}^*}{\partial c_j} \right|_{a^*} + \frac{\partial a^*}{\partial c_j} \frac{d\gamma_{\text{false}}^*}{da^*} = 0. \quad (29)$$

Using Eqs. (28) and (26), and cancelling an overall $-ba^{*j+1}$ factor, this would reduce to

$$j - \sum_i i(i+1)c_i a^{*i} / \sum_i i c_i a^{*i} = 0. \quad (30)$$

But this equation would have to be true for all $j = 2, 3, \dots$, which is clearly impossible since the second term is independent of j .

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